

# On invariant sets of diffeomorphisms

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## Abstract

We give a simple upper bound for the upper box dimension of a backward invariant set of a  $C^1$ -diffeomorphism of a Riemannian manifold. We also estimate an upper bound for the box dimension of a forward invariant set of a  $C^1$ -mapping with finite Brouwer degree in a Riemannian manifold.

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*Key words:* Riemannian manifold, Invariant set, Box dimension.

## 1. Introduction

The direct computation of the Hausdorff dimension of invariant compact sets is a problem of high complexity. Therefore, it is interesting to obtain analytic estimates of this dimension. Recently, Many research studies have been developed on the investigation of the Hausdorff dimension of invariant sets of discrete dynamical systems. First results in this direction are given in [3] for compact subsets of  $\mathbb{R}^n$  that are backward invariant under  $C^1$ -maps. Wolf in [24] gave Hausdorff dimension estimates (related to the values and behavior of  $\det D_x f$  and  $\|D_x f\|$ ) for compact forward invariant sets of  $C^1$ -diffeomorphisms in  $\mathbb{R}^n$ . In [13, 14] the conditions of these estimates are weakened using a Lyapunov type function in  $\mathbb{R}^n$ . In [21], Temam gave upper bounds for the fractal dimension of flow-invariant sets in a Hilbert space, which is proved in [10] for vector fields on Riemannian manifolds. In [7], Franz considered compact invariant sets of  $C^1$ -diffeomorphisms for which there exists an equivalent splitting of the tangent bundle. Qu and Zhou in [2] generalized the results of [24] to the map on smooth Riemannian manifolds with non-negative Ricci curvature. In [18], Wolf's theorem is generalized to complete Riemannian manifolds without conditions on curvature. For further studies of estimation of upper bounds for Hausdorff dimension of invariant compact sets, one may consult [4, 12, 15, 20, 23]. In the present paper, we estimate an upper bound for Hausdorff and box dimension of compact backward invariant sets of  $C^1$ -diffeomorphisms on Riemannian manifolds with conditions on  $\min S_n(D_x f)$  and  $\max |\det D_x f|$ . We

also estimate an upper bound for the box dimension of a compact forward invariant set of a  $C^1$ -mapping with finite Brouwer's degree on Riemannian manifolds.

## 2. Preliminaries

We will use the following definitions and facts.

(1) Consider a linear operator  $L : E \rightarrow E'$  between two Euclidean spaces of dimension  $n$  with scalar products  $\langle \cdot, \cdot \rangle_E$  and  $\langle \cdot, \cdot \rangle_{E'}$  respectively. The adjoint operator of  $L$  is the unique linear operator  $L^* : E' \rightarrow E$ , determined by  $\langle Lx, y \rangle_{E'} = \langle x, L^*y \rangle_E$  for all  $x \in E$  and  $y \in E'$ . The eigenvalues of the positive semi-definite operator  $\sqrt{L^*L}$  are the singular values of  $L$ . The singular values are all non-negative, usually listed in order to their size and multiplicity  $S_1(L) \geq S_2(L) \geq \dots \geq S_n(L)$ . The absolute value of the determinant of  $L$  is stated as the square root of the determinant of  $L^*L$ .

(2) Let  $M, N$  be Riemannian manifolds of dimension  $n$ ,  $U$  be an open subset of  $M$ .

(a) If  $f : U \rightarrow N$  is a  $C^1$ -diffeomorphism. We denote the tangent map of  $f$  at the point  $x \in M$  by  $D_x f : T_x M \rightarrow T_{f(x)} N$  and the norm of  $f$  at that point, is defined by

$$\|D_x f\| = \sup\{|D_x f(v)| : v \in T_x M; |v| = 1\}$$

(b) For  $r > 0$  the  $r$ -neighborhood of a set  $F \subset M$  is defined by

$$B_r(F) = \{x \in M : d(x, a) < r \text{ for some } a \in F\}$$

(c) Let  $F$  be a non-empty bounded subset of  $M$  and  $N_\delta(F)$  be the smallest number of balls of radius at most  $\delta$  which can cover  $F$ . The upper box dimension of  $F$  is defined (see [5]) by

$$\overline{\dim}_B F = \limsup_{\delta \rightarrow 0} \frac{\log(N_\delta F)}{-\log \delta}$$

(d) If  $M$  and  $N$  are compact orientable manifolds and  $f : M \rightarrow N$  is a differentiable map and  $y \in N$  is a regular value of  $f$ , then the Brouwer degree of  $f$  at  $y$  is defined (see [16]) by

$$\deg(f) = \sum_{x \in f^{-1}(y)} \text{sgn}(D_x(f))$$

Where,  $\text{sgn}(D_x(f))$  equals  $+1$  or  $-1$  according to  $D_x(f)$ , which it preserves or reverses orientation.

(e) If  $f : U \rightarrow M$  is a  $C^1$ -diffeomorphism onto its image. A compact subset  $K$  of  $U$  is called forward  $f$ -invariant if  $f(K) \subset K$ . If  $K \subset f(K)$  then  $K$  is called backward  $f$ -invariant.

Authors of [1] gave a fractal dimension estimate for the invariant set of a function  $f : U \subset M \rightarrow M$  under the conditions

$$0 < \min S_n(D_x f) < \sqrt{n}^{-1}$$

and

$$(\max |det D_x f|)(\min S_n(D_x f))^{d-n} \leq 8^{-n} n^{\frac{-d}{2}}.$$

Qu and Zhou in [2] weakened the conditions and upgraded the results of [1] under the condition that Ricci curvature is non-negative. In this paper we generalize the results of [1] to complete Riemannian manifolds (without considerations on Ricci curvature). We will prove the following theorems.

**Theorem 1.1.** *Let  $U \subset M$  be an open subset of a  $n$ -dimensional Riemannian manifold  $M$ ,  $f : U \rightarrow M$  be a  $C^1$ -diffeomorphism onto its image and  $K$  be a backward  $f$ -invariant set. If  $0 < \min S_n(D_x f) < 1$  and there exists a number  $d \in (0, d]$  such that*

$$(\max |\det D_x f|)(\min S_n(D_x f))^{d-n} \leq 1,$$

then

$$\overline{\dim}_B K \leq d.$$

The following theorem is proved in [24] under the condition  $M = \mathbb{R}^n$ . We prove the same result in more general case, when  $M$  is a complete Riemannian manifold.

**Theorem 1.2.** *Let  $U \subset M$  be an open subset of a complete Riemannian manifold  $M$  and  $f : U \rightarrow M$  be a  $C^1$ -mapping with the Brouwer degree  $d$ . Let  $K \subset U$  be a compact  $f$ -invariant set and suppose that  $f$  has a non-zero Jacobian determinant. Put*

$$b = \lim_{m \rightarrow \infty} \frac{1}{m} \log(\min\{|\det D_x f^m|, x \in K\})$$

$$s = \lim_{m \rightarrow \infty} \frac{1}{m} \log(\max\{\|D_x f^m\|, x \in K\})$$

If  $b > 0$ , then  $s > 0$  and

$$\overline{\dim}_B K \leq n - \frac{b - \log d}{s} < n$$

**Remark 1.3.** Because the Huasdorff dimension of a set is smaller or equal to its upper box dimension, theorems 1.1 and 1.2 also give upper bounds for the Huasdorff dimension of  $K$ .

**Remark 1.4.** In Theorem 1.1, the assumption  $0 < \min S_n(D_x f) < 1$  would be unnecessary provided that the inequality  $(\max |\det D_x f|)(\min S_n(D_x f))^{d-n} \leq 1$  was strict.

## 2. Proofs of the theorems

**Lemma 2.1** (see [18]). *If  $K$  is a compact subset of a Riemannian manifold  $M$  and  $\dim M = n$ , then*

$$\overline{\dim}_B K \leq n + \limsup_{r \rightarrow 0} \frac{\log(\text{vol}(B_r K))}{-\log(r)}$$

**Fact 2.2** (see [8, Theorem 2.92]). If  $M$  is a Riemannian manifold and  $x_0 \in M$ , then there is an open ball around  $x_0$  such that for any  $x, y \in U$  there is a unique geodesic  $\gamma$  joining  $x$  to  $y$  with the length equal to  $d(x, y)$ .

**Remark 2.3** (see [2]). Let  $B \subset U$  be an open subset of a Riemannian manifold  $M$  and  $f : U \rightarrow M$  a  $C^1$ -map. If  $B$  is bounded then

$$\text{vol}(f(B)) \geq \inf_{x \in B} |\det D_x f| \text{vol}(B).$$

**Remark 2.4.** If  $U$  is an open subset of a Riemannian manifold  $M$  and  $f : U \rightarrow M$  a  $C^1$ -diffeomorphism on its image, It is proved in [18] that if  $K \subset U$  is a compact forward  $f$ -invariant set and

$$b = \lim_{m \rightarrow \infty} \frac{1}{m} \log(\min\{|\det D_x f^m|, x \in K\})$$

$$s = \lim_{m \rightarrow \infty} \frac{1}{m} \log(\max\{\|D_x f^m\|, x \in K\})$$

as well as  $b > 0$ , then  $s > 0$  and

$$\overline{\dim}_B K \leq n - \frac{b}{s} < n$$

In a similar way we can prove the following theorem.

**Theorem 2.5.** *Let  $U$  be an open subset of a Riemannian manifold  $M$  and*

$f : U \rightarrow M$  a  $C^1$ -diffeomorphism on its image. Let  $K \subset U$  be a compact backward  $f$ -invariant set. Define

$$b = \lim_{m \rightarrow \infty} \frac{1}{m} \log(\min\{| \det D_x f^{-m} |, x \in K\})$$

$$s = \lim_{m \rightarrow \infty} \frac{1}{m} \log(\max\{\|D_x f^{-m}\|, x \in K\})$$

If  $b > 0$ , then  $s > 0$  and

$$\overline{\dim}_B K \leq n - \frac{b}{s} < n$$

### Proof of Theorem 1.1.

Since all norms in  $\mathbb{R}^n$  are equivalent, the values of  $b$  and  $s$  are independent of the norm. Therefore the norm of  $D_x f : T_x M \rightarrow T_{f(x)} M$  is equal to

$$\|D_x f\| = \sqrt{\alpha_n}$$

Where  $\alpha_1$  is the maximum eigenvalue of  $|D_x f|^t D_x f$ . Thus we have

$$(1) \quad S_n((D_x \varphi)^{-1}) = \|D_x \varphi\|^{-1}$$

By (1) and the assumption that there exists a number  $d \in (0, d]$  such that  $(\max|\det D_x f|)(\min S_n(D_x f))^{d-n} \leq 1$ , we have

$$(2) \quad \max|\det(D_x f)| \leq (\min S_n(D_x f))^{n-d} = ((\max\|(D_x f)^{-1}\|)^{-1})^{n-d}$$

Using  $f^{-1}(K) \subset K$  and (2) we have

$$(\max\|(D_x f)^{-1}\|)^{n-d} \leq \min|\det(D_x f^{-1})|$$

Furthermore,

$$\min|\det(D_x f^{-m})| = \min|\det(D_x f^{-1}) \dots \det(D_{f^{-m+1}(x)} f^{-1})| \geq (\min|\det(D_x f^{-1})|)^m$$

And

$$\max\|D_x f^{-m}\| \leq \max(\|D_x f^{-1}\| \dots \|D_{f^{-m+1}(x)} f^{-1}\|) \leq (\max|\det(D_x \varphi^{-1})|)^m$$

By the assumptions of Theorem 1.1 we have

$$\max|\det(D_x f)| \leq (\min S_n(D_x f))^{n-d} \leq \min S_n(D_x f)$$

Which results

$$\text{mindet}(D_x f^{-1}) > 1$$

Therefore,

$$\begin{aligned} \frac{b}{s} &= \lim_{m \rightarrow \infty} \frac{\log(\min | \det D_x \varphi^{-m} | : x \in k)}{\log(\max \| D_x \varphi^{-m} \| : x \in k)} \geq \\ &\geq \lim_{m \rightarrow \infty} \frac{\log(\min | \det D_x \varphi^{-1} |)^m}{\log(\max \| D_x \varphi^{-1} \|)^m} \geq n - d \end{aligned}$$

Now by Theorem 2.5 we have  $\overline{\dim}_B K < d$ .

□

### Proof of Theorem 1.2.

Since  $f$  is a  $C^1$ -mapping, it follows from the definition of  $b$  and  $s$  and the continuity argument that for each  $\delta > 0$ , there exists  $K_\delta \in \mathbb{N}$  and  $\epsilon > 0$  such that

$$(3) \quad 1 < \exp(k_\delta(b - \delta)) < |\det D_x f^{k_\delta}|$$

and

$$(4) \quad \|D_x f^{k_\delta}\| < \exp(k_\delta(s + \delta))$$

for all  $x \in B_\epsilon(K)$ . Since the Jacobian determinant is non-zero on  $K$ , there exists a neighborhood of  $K$  on which  $f$  can be considered locally as a  $C^1$ -diffeomorphism onto its image. Let us assume  $B_\epsilon(K)$  is such a neighborhood. It is possible to choose  $\epsilon$  sufficiently small that for each  $x \in K$  and each positive number  $\varsigma \leq \epsilon$ ,  $B_\varsigma(x)$  admits the results of Fact 2.2. From now on consider the mapping  $g = f^{k_\delta}$ . Notice that  $K$  is also backward  $g$ -invariant. Put

$$r_m = \epsilon (\exp(k(s + \delta)))^{-m} < \epsilon$$

and

$$B_m = B_{r_m}(K)$$

for all  $m \in \mathbb{N}$ .

(1) Let  $x \in B_1$ , then there exists  $y \in K$  such that  $d(x, y) < r_1$ . By Fact 2.2, there is a minimal geodesic  $\gamma : [0, 1] \rightarrow B_1(x)$  from  $x$  to  $y$ . So

$$d(g(x), g(y)) \leq \int_0^1 \left| \frac{d}{dt} g(\gamma(t)) \right| dt \leq \int_0^1 \|D_{\gamma(t)} g\| \cdot \left| \frac{d}{dt} \gamma(t) \right| dt \leq$$

$$\begin{aligned}
&\leq (\exp(k_\delta(s + \delta))) \int_0^1 \left| \frac{d}{dt} \gamma(t) \right| dt = (\exp(k_\delta(s + \delta))) d(x, y) < \\
&< (\exp(k_\delta(s + \delta))) \cdot r_1 = \epsilon
\end{aligned}$$

(2) Now let  $x \in B_2$ , then there exists  $y \in K$  such that  $d(x, y) < r_2$ . Similarly, we have

$$\begin{aligned}
d(g^2(x), g^2(y)) &\leq \int_0^1 \left| \frac{d}{dt} g^2(\gamma(t)) \right| dt \leq \\
&\leq \int_0^1 \|D_{\gamma(t)} g^2\| \cdot \left| \frac{d}{dt} \gamma(t) \right| dt = \int_0^1 \|D_{\gamma(t)} g\| \cdot \|D_{g(\gamma(t))} g\| \cdot \left| \frac{d}{dt} \gamma(t) \right| dt \leq \\
&\leq (\exp(k_\delta(s + \delta)))^2 \int_0^1 \left| \frac{d}{dt} \gamma(t) \right| dt = (\exp(k_\delta(s + \delta)))^2 d(x, y) < \\
&< (\exp(k_\delta(s + \delta)))^2 \cdot r_2 = \epsilon
\end{aligned}$$

By induction, for any  $m \in \mathbb{N}$  and any  $x \in B_m$ , there exists  $y \in K$  such that  $d(g^m(x), g^m(y)) \leq \epsilon$ . Since  $g^m(K) \subset K$  then  $g^m(x) \in B_\epsilon K$ . Thus  $g^m(B_m(K)) \subset B_\epsilon K$  for all  $m \in \mathbb{N}$ . Which results,

$$(5) \quad \text{vol}(g^m(B_m(K))) \leq \text{vol}(B_\epsilon(K))$$

$B_m(K)$  is bounded, so there exist balls  $U_{k_1}, \dots, U_{k_n}$  such that  $g^m|_{U_{k_1}}$  is a  $C^1$ -diffeomorphism onto its image and  $B_\epsilon(K) \subset \bigcup_{i=1}^n U_{k_i}$ . Put

$$\begin{aligned}
V_{k_1} &= U_{k_1} \cap B_\epsilon(K) \\
V_{k_i} &= (U_{k_i} \cap B_\epsilon(K)) \setminus \bigcup_{s=1}^{i-1} V_{k_s}
\end{aligned}$$

By Remark 2.3 we have,

$$(6) \quad \text{vol}(g^m(V_{k_i})) \geq \inf_{x \in V_{k_i}} |\det D_x g^m| \text{vol}(V_{k_i})$$

By (3), We have  $\exp(K_\delta(b - \delta)) < |\det D_x g|$ , thus  $\exp(K_\delta(b - \delta))^m < |\det D_x g^m|$ . Therefore by (6), we have

$$(7) \quad \text{vol}(V_{k_i}) \leq \exp(K_\delta(b - \delta))^{-m} \text{vol}(g^m(V_{k_i}))$$

The sets  $V_{k_i}$  are pairwise disjoint and  $\bigcup_1^n V_{k_i} = B_\epsilon(K)$ . Using (5) and (7) we get

$$\text{vol}(B_\epsilon(K)) = \sum_{i=1}^n \text{vol}(V_{k_i}) \leq$$

$$\begin{aligned}
&\leq \sum_{l=1}^n \exp(K_\delta(b-\delta))^{-m} \text{vol}(g^m(V_{k_i})) \leq \\
&\leq d^{km} \exp(K_\delta(b-\delta))^{-m} \text{vol}(B_\epsilon(K)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\limsup_{r \rightarrow 0} \frac{\log(\text{vol}(B_r(K)))}{-\log(r)} = \limsup_{r_m \rightarrow 0} \frac{\log(\text{vol} B_m(K))}{-\log(r_m)} \leq \\
&\leq \lim_{m \rightarrow \infty} \frac{\log(d^{km} \exp(K_\delta(b-\delta))^{-m} \text{vol}(B_\epsilon(K)))}{-\log(\frac{\epsilon}{\exp(k_\delta(s+\delta))^m})} = -\frac{b-\delta-\log d}{s+\delta}
\end{aligned}$$

Since  $\delta$  is arbitrary small, then

$$(8) \quad \limsup_{r \rightarrow 0} \frac{\log(\text{vol}(B_r(K)))}{-\log(r)} \leq -\frac{b-\log d}{s}$$

Now by (8) and Lemma 2.1, we get

$$\overline{\dim}_B K \leq n - \frac{b-\log d}{s}$$

□

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